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Ordering of some boson operator functions

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Abstract. The problem of normal ordering of the following boson operator functions $(a+a^+)^m$, $(a^r+a^+)^m$, $(a+N)^m$, $(a^2+N)^m$, where $N=a^+a$, is solved explicitly. Simple algebraic methods are used. As an intermediate result the ordering formulae for the symmetrised products of boson operators are presented. There is agreement with earlier results of other authors and generalisation of these results. A new representation of the generalised Stirling numbers used for the ordering of the binomial $(a^2+N)^m$ is given and several principal properties of these numbers are derived.

1. Introduction

There is a rather elaborate apparatus for the ordering of the boson operator functions (e.g. Wilcox 1967, Klauder and Sudarshan 1968, Agarwal and Wolf 1970, Louisell 1973). However, there is a lack of concrete formulae. In spite of the fact that this apparatus, in principle, gives an answer to a great number of different problems, the derivation of explicit formulae presents many difficulties. Our aim is to obtain several new normal-ordering formulae for some functions with the help of simple algebraic methods.

Let us consider creation and annihilation boson operators a^+ and a with the commutator

$$[a, a^+]=c \tag{1}$$

where c is a so-called c number. Witschel (1975) pointed out that Yamazaki (1952) suggested a tentative formula

$$(a+a^+)^m=\sum_{k=0}^{[m/2]}\sum_{i=0}^{m-2k}\frac{c^k m!(a^+)^i a^{m-2k-i}}{2^k k!i!(m-2k-i)!} \tag{2}$$

The bracket symbol $[m/2]$ means the integer less than or equal to $m/2$. Lately Cohen (1966) and Wilcox (1967) derived the same formula. In this paper we found another kind of the same ordering formula with the help of our method. This method also leads us to the ordering of the function $(a^++a^2)^m$ and of the general function $(a^++a^r)^m$.

Another of our results is the generalisation of the Katriel (1974) formula

$$N^l=\sum_i S(l,i)c^{l-i}(a^+)^i a^i \tag{3}$$

where

$$S(l,i)=\frac{1}{i}\sum_{j=0}^{i}\binom{i}{j}j^l(-1)^{i-j} \tag{4}$$

are the Stirling numbers of the second kind (Riordan 1958) and $N=a^+a$. Our generalisation extends to the ordering of two functions $(a+N)^m$ and $(a^2+N)^m$.

Moreover, we give here as intermediate results the ordering formulae for the symmetrised products of boson operators such as $\{(a^+)^k, a^l\}$, $\{(a^+)^k, (a^2)^l\}$, $\{(a^+)^k, (a^r)^l\}$, $\{a^k, N^l\}$, $\{(a^2)^k, N^l\}$, where k and l are arbitrary positive integers. In some cases these expressions have independent values, e.g. $\{(a^+)^k, a^l\}$ is the operator eigenpolynomial creating the unnormalised angular momentum state $|j, m\rangle$, where $j=\frac{1}{2}(k+1)$, $m=\frac{1}{2}(k-1)$ in the symplecton boson realisation of SU(2) by Biedenharn and Louck (1971). The above-mentioned binomials may find applications in quantum optics when the interaction of a quantised radiation field with a quantum-mechanical system of atoms is considered (e.g. Solovarov 1980).

In the case of the ordering of $\{(a^2)^k, N^l\}$ we express our solution by means of the generalised Stirling numbers and give a new representation of these numbers as a convolution of the Stirling numbers of the first and second kinds. From one basic definition we derive several principal properties of the generalised Stirling numbers such as generating functions, recurrence relations, convolution with the inverse numbers etc. A description of these numbers is given in the appendix.

2. Symmetrisers

We define a symmetriser as a summation of the products of all possible permutations of k operators A and l operators B

$$\{A^k, B^l\} \equiv \{B^l, A^k\} = A^k B^l + A^{k-1}BAB^{l-1} + A^{k-2}BA^2B^{l-1} + \ldots . \tag{5}$$

A and B are non-commutative operators. The number of terms in these symmetrisers is equal to

$$M(k,l) = \binom{k+l}{k}. \tag{6}$$

The following relation between the binomial $(A+B)^m$ and the symmetrisers is evident:

$$(A+B)^m = \sum_{k=0}^{m} \{A^k, B^{m-k}\}. \tag{7}$$

We do not give a special proof of (7) because of its simplicity, and note only that the equality of the numbers of terms on both sides of (7) follows from

$$2^m = \sum_{i=0}^{m} \binom{m}{i}.$$

One sees that the ordering problem for the binomial is reduced to the same problem for the symmetriser. To solve the latter we will use the easily understood relations

$$\{A^k, B^l\} = A\{A^{k-1}, B^l\} + B\{A^k, B^{l-1}\}, \tag{8}$$

$$\{(\alpha A)^k, (\beta B)^l\} = \alpha^k \beta^l \{A^k, B^l\}, \tag{9}$$

$$\{A^k, B^0\} = A^k, \qquad \{A^0, B^l\} = B^l, \tag{10}$$

where α and β are arbitrary numbers. The following elementary property of the binomial coefficients reflects the numbers of terms in the left- and right-hand parts of (8):

$$\binom{m-1}{k-1}+\binom{m-1}{k}=\binom{m}{k}, \qquad m=l+k. \tag{10a}$$

Below we also use the commutator (Shalitin and Tikochinsky 1979)

$$[a^l,(a^+)^k]=\sum_{i=1}^{\min(k,l)} i!\binom{l}{i}\binom{k}{i}c^i(a^+)^{k-i}a^{l-i}.$$

3. Ordering formulae

3.1. $A=a^+, B=a$

In this case we have obtained the ordering of the symmetriser in the form

$$\{(a^+)^k,a^l\}=\sum_{i=0}^{\min(k,l)}\frac{(k+l)!}{(2i)!!(k-i)!(l-i)!}c^i(a^+)^{k-i}a^{l-i}. \tag{11}$$

Using equation (8) we can prove this relation by induction. Also it was possible to get the inverse formula

$$(a^+)^ka^l=\sum_{i=0}^{\min(k,l)}\frac{(-c)^ik!l!}{(2i)!!(k+l-2i)!}\{(a^+)^{k-i},a^{l-i}\}. \tag{12}$$

We can now check (11) and (12) by substituting one into the other and using the simple relation $\Sigma_{i=0}^{k}(-1)^i\binom{k}{i}=0$. Substituting (11) into (7) we obtain the formula

$$(a+a^+)^m=\sum_{k=0}^{m}\sum_{i=0}^{\min(k,m-k)}\frac{c^im!}{2^ii!(k-i)!(m-k-i)!}(a^+)^{k-i}a^{m-k-i}, \tag{13}$$

which is another kind of the Yamazaki formula (2). In equation (2) each ordered homogeneous polynomial of operators $(a^+)^ia^{r-i}$ of the power $r=m-2k$ is labelled by a different number k, but in (13) it is the symmetriser, non-homogeneous ordered polynomial, which corresponds to the definite k. The formula (13) can be transformed into the formula (2) by interchanging the order of summation and substituting $l+i$ for k.

3.2. $A=a^+, a; B=a^2, (a^+)^2$

The ordered forms of the symmetrisers in this case were found to be

$$\begin{bmatrix}\{(a^+)^k,(a^2)^l\}\\ \{a^k,(a^{+2})^l\}\end{bmatrix}=\sum_{n=0}^{\min(k,2l)}\frac{(k+l)!}{(k-n)!}P^2_{nl}c^n\begin{bmatrix}(a^+)^{k-n}a^{2l-n}\\ (a^+)^{2l-n}a^{k-n}\end{bmatrix} \tag{14}$$

where

$$P^2_{nl}=\sum_i\frac{1}{(l-i)!(2i-n)!3^{n-i}(n-i)!}. \tag{15}$$

Our proof of these formulae consists of two steps. Firstly, we assume that (14) is true and obtain the recurrence relations for coefficients P

$$(n+l)P^2_{nl}=P^2_{n,l-1}+2P^2_{n-1,l-1}+P^2_{n-2,l-1}. \tag{16}$$

Secondly, the explicit form of these coefficients which would satisfy both initial conditions and these relations (16) has been found. The solution is (15).

Two cases considered in §§ 3.1 and 3.2 have been written out separately because of their frequent use in practice. Another reason is that the more general case from § 3.3 is being solved with the help of rather lengthy expressions.

3.3. $A = a^+, B = a^r$; *r is a positive integer*

According to the scheme of § 3.2 it has also become possible to find here an explicit solution. The recurrence relation (8) is satisfied by an ordering formula similar to (14)

$$\{(a^+)^k, (a^r)^l\} = \sum_{n=0}^{\min(k,rl)} \frac{(k+l)!}{(k-n)!} P^r_{nl} c^n (a^+)^{k-n} a^{rl-n}. \tag{17}$$

We obtain the ordering of $\{a^k, (a^{+r})^l\}$, carrying the analogy from (14) to (17). The coefficients P must then satisfy the condition

$$(n+l)P^2_{nl} = \sum_{j=0}^{r} \binom{r}{j} P^r_{n-j,l-1} \tag{18}$$

which, as can be easily seen, transforms into (16) for $r = 2$. The explicit form of the coefficients P is

$$P^r_{nl} = \sum_{i_1, i_2, \ldots, i_{r-1}} \frac{1}{(l-i_1)!(\Sigma i + i_{r-1} - n)!(n - \Sigma i)!(r+1)^{n-\Sigma i}} \times \prod_{k=1}^{r-2} \frac{1}{(i_k - i_{k+1})!} \left[\frac{1}{k+1}\binom{r}{k}\right]^{i_k - i_{k+1}}, \qquad \sum i = i_1 + i_2 + \ldots i_{r-1}; \tag{19}$$

$$\begin{aligned} &i_1 = [(n+r-1)/r], [(n+r-1)/r]+1, \ldots, n; \\ &i_2 = [(n+r-i_1-2)/(r-1)], [\ ]+1, \ldots, \min(n-i_1, i_1); \\ &i_3 = [(n+r-i_1-i_2-3)/(r-2)], [\ ]+1, \ldots, \min(n-i_1-i_2, i_2); \\ &i_4 = [(n+r-i_1-i_2-i_3-4)/(r-3)], [\ ]+1, \ldots, \min(n-i_1-i_2-i_3, i_3); \ldots. \end{aligned} \tag{20}$$

Conditions (20) giving the limits of alteration for the integers i_k are another form of the simple condition: all the expressions under the factorial sign in (19) must be equal to or greater than zero: $l - i_1 \geqslant 0$, $\Sigma i + i_{r-1} - n \geqslant 0$, $n - \Sigma i \geqslant 0$, $i_k - i_{k+1} \geqslant 0$. Conditions (20) prove to be more convenient in practice. The fact that coefficients P from (19) satisfy the relations (18) can be generally proved with the help of direct substitution. The following simple identity must be used here:

$$(n+l) \equiv l - i_1 + \sum_{k=1}^{r-2} (k+1)(i_k - i_{k+1}) + r\left(\sum i + i_{r-1} - n\right) + (r+1)\left(n - \sum i\right)$$

where $r = 1, 2, 3, \ldots$. This formula holds true for $r = 1, 2$ but in order to avoid different readings let us write out these particular cases separately.

$$n + l \equiv l - n + 2n, \qquad r = 1,$$

$$n + l \equiv l - i + 2(2i - n) + 3(n - i), \qquad r = 2.$$

So (17) and (19) give the explicit form of ordered symmetrisers which was discussed and therefore of ordered binomials $(a^+ + a^r)^m$ and $(a + a^{+r})^m$.

3.4. $A = a; B = N$

The ordered symmetrisers in this case were found to be

$$\{a^k, N^l\} = \sum_{i=0}^{l} \binom{k+i}{i} S(k+l, k+i) c^{l-i} (a^+)^i a^{k+i}. \tag{21}$$

Let us verify this equation. By means of induction we obtain

$$\begin{aligned}\{a^k, N^l\} &= a\{a^{k-1}, N^l\} + N\{a^k, N^{l-1}\} \\ &= \sum_i \left[\binom{k+i-1}{i} S(k+l-1, k+i-1) + \binom{k+i}{i}(k+i) S(k+l-1, k+i) \right. \\ &\quad \left. + \binom{k+i-1}{i-1} S(k+l-1, k+i-1)\right] c^{l-i} (a^+)^i a^{k+i}. \end{aligned} \tag{22}$$

After the use of recurrence relations for the binomial coefficients (10*a*) and for the Stirling numbers of the second kind

$$S(k+1, l) = S(k, l-1) + lS(k, l) \tag{23}$$

the square bracket in (22) is reduced to the form

$$\binom{k+i}{i} S(k+l, k+i)$$

which gives the desired proof of (21). Taking into account equation (10) we can easily see that in the case $k = 0$ equation (21) is reduced to the Katriel formula (3).

3.5. $A = a^2, B = N$

The corresponding ordering formula is

$$\{(a^2)^k, N^l\} = \sum_{m=0}^{l} B_m^{kl} c^m (a^+)^{l-m} a^{2k+l-m}, \tag{24}$$

If we substitute (24) in (8) we get the recurrence relation for coefficients B

$$B_m^{kl} = B_m^{k-1,l} + B_m^{k,l-1} + (2k+l-m) B_{m-1}^{k,l-1}, \qquad B_0^{10} = B_0^{01} = 1. \tag{25}$$

For $k = 0$ relation (25) becomes the recurrence relation for the Stirling numbers of the second kind (23); for $m = 0$ (25) becomes the recurrence relation for the binomial coefficients and therefore we have

$$B_m^{0l} = S(l, l-m), \qquad B_0^{kl} = \binom{k+l}{l}. \tag{26}$$

So the case $k = 0$ converts into the early result (3). The general solution of (25) is

$$B_m^{kl} = \sum_i \binom{k+l}{k+2i} S(k+l-2i, k+l-m, k) D_i^k \tag{27}$$

where

$$S(n, r, k) = \sum_{i=0}^{\infty} s(k, k-i) S(n-i, r) \tag{28}$$

are a new form of the generalised Stirling numbers. These numbers are a generalisation of the Stirling numbers of the first and second kind. We will give a discussion of their properties in the appendix.

As far as the numbers D_i^k from (27) are concerned, they both have two recurrence relations

$$D_i^k = k^2 D_{i-1}^k + D_i^{k-2}, \tag{29}$$

$$D_i^k = \sum_{j=0}^{i} k^{2(i-j)} D_j^{k-2}, \tag{30}$$

and two different explicit forms

$$D_i^k = \sum_{j,r,s,\ldots t} k^{2j}(k-2)^{2r}(k-4)^{2s}\ldots \begin{bmatrix} 2^{2t} \\ 1 \end{bmatrix}, \qquad \begin{matrix} k\text{—even integer,} \\ k\text{—odd integer,} \end{matrix}$$

$$j+r+s+\ldots+t=i, \tag{31}$$

$$D_i^k = 2^{1-k}\sum_r (-1)^r (k-2r)^{k+2i}/r!(k-r)!,$$

$$r=0,1,2,\ldots,(k-1)/2, \qquad k=0,1,2,\ldots. \tag{32}$$

We must also add several initial values to these formulae

$$D_i^0 = \delta_{0i}, \qquad D_i^1 = D_0^k = 1, \qquad D_i^2 = 4^i.$$

By means of simple calculations it is possible to show that D_i^k from (32) satisfy (29), that (31) is a consequence of (30) and (30) follows from (29). The right-hand expression from (32) can be considered as a solution of the summation problem (31).

It is not difficult to prove that B_m^{kl} from (27) satisfy the recurrence relation (25). We do not give this cumbersome calculation. In some particular cases yet we may easily check the correctness of (27). For example, taking into account that $D_i^0 = \delta_{0i}$, $S(n, r, 0) = S(n, r)$ we get $B_m^{01} = S(l, l-m)$ for $k=0$ as one would expect, see (26). For the other case $m=0$, taking into account that $S(n, r, k)=0$ for $n<r$ and $S(n, n, k)=1$, we can easily reduce (27) to (26). For $l=m$ we get a new representation of the Stirling numbers of the second kind

$$S(k+m, k) = 2^{-m}\sum_i \binom{m+k}{k+2i} k^{m-2i} D_i^k. \tag{33}$$

To derive the last formula we consider (25); for $l=m$ it transforms to the recurrence relation

$$B(k, m) = B(k-1, m) + 2kB(k, m-1)$$

where $B(k, m) \equiv B_m^{km}$, which has a solution in the form

$$B(k, m) = 2^m S(k+m, k).$$

Substituting these numbers into the left-hand side of equation (27), we obtain (33).

At the end of § 3.5 it may be said that formulae (27), (28) and (32) (or (30)) present an explicit form of the coefficients B and consequently our problem is solved.

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Appendix. Generalised Stirling numbers

We have used generalised Stirling numbers (28) for the ordering of the bionomial $(a^2+N)^m$. Here, we will give their definition and derive their basic properties. Generalised Stirling numbers have been known for some time (e.g. Medvedev and Ivchenko 1965, Egorichev 1977), yet the formulae given below appear to be published for the first time. Our description has characteristic properties such that:

(1) all indices in all formulae, when there are not special restrictions, can be any integer, positive or negative;

(2) the numbers present a generalisation of both Stirling numbers of the first kind and Stirling numbers of the second kind.

The Stirling numbers of the first and second kinds, (4), defined by Riordan (1958)

$$(x)_n = \sum_r s(n, r)x^n, \tag{A1}$$

$$x^n = \sum_r S(n, r)(x)_r, \tag{A2}$$

where

$$(x)_n = x(x-1)(x-2)\ldots(x-n+1),$$

$$(x)_{-n} = [(x+1)(x+2)\ldots(x+n)]^{-1}, \qquad (x)_0 = 1, \qquad (x)_1 = x,$$

are used as a starting point.

(1) Our definition of generalised Stirling numbers is (28):

$$S(n, r, k) = \sum_{i=0}^{\infty} s(k, k-i)S(n-i, r). \tag{A3}$$

We assume that the inverse matrices $S^{-1}(n, r, k)$ exist, so that

$$\sum_r S(n, r, k)S^{-1}(r, m, k) = \sum_r S(r, n, k)S^{-1}(m, r, k) = \delta_{m,n}. \tag{A4}$$

(2) On the basis of (A3) and (A4) we can construct four generating functions

$$\sum_r S(n, r, k)(x)_r = x^{n-k}(x)_k, \tag{A5}$$

$$\sum_n S(n, r, k)x^{r-n} = x^{r-k+1}/(x-k)_{r-k+1} = x^{r-k+1}(x-r-1)_{k-r-1}, \tag{A6}$$

$$\sum_r S^{-1}(n, r, k)x^r = x^k(x-k)_{n-k}, \tag{A7}$$

$$\sum_n S^{-1}(n, r, k)/(x-k)_{n-k+1} = x^{k-r-1}. \tag{A8}$$

(A5) can be proved by substituting (A3) in to the left-hand side of (A5), interchanging the order of summation and successively using (A2) and (A1). To prove (A6) the generating function

$$\sum_n S(n, r)x^{r-n} = x^{r+1}/(x)_{r+1} \tag{A9}$$

is required. In an analogous generating function of Abramowitz and Stegun (1965) we replace x by $1/x$ and now have a formula which is correct for all positive and negative integers r. We substitute (A3) in the left-hand side of (A6) and with the help of (A9) and (A1) obtain the equation (A6). Equations (A7) and (A8) can be deduced from (A5) and (A6) by using the orthogonality relation (A4).

(3) Then we note the resemblance of the right-hand sides of (A6) and (A7). Making the substitutions $n = N+K-R-1$, $r = K-1$, $k = N$ in (A6) we obtain an expression the right-hand side of which is equal to the right-hand side of (A7). Consequently

$$S^{-1}(n, r, k) = S(n+k-r-1, k-1, n), \tag{A10}$$

$$S(n, r, k) = S^{-1}(k, r+k-n, r+1). \tag{A11}$$

(4) Recurrence relations for generalised Stirling numbers

$$S(n, r, k) = S(n, r, k-1) - (k-1)S(n-1, r, k-1), \tag{A12}$$

$$S(n, r, k) = S(n-1, r-1, k) + rS(n-1, r, k), \tag{A13}$$

$$S^{-1}(n, r, k) = S^{-1}(n, r, k-1) + (k-1)S^{-1}(n, r+1, k), \tag{A14}$$

$$S^{-1}(n, r, k) = S^{-1}(n-1, r-1, k) - (n-1)S^{-1}(n-1, r, k) \tag{A15}$$

can be derived from generating functions (A5)–(A8) and relations (A10)–(A11). For example, applying the simple recurrence formula

$$(x)_k = x(x)_{k-1} - (k-1)(x)_{k-1} \tag{A16}$$

to the right-hand side of (A5) we can easily obtain (A12). Applying (A16) to (A5) in another way

$$\sum_r S(n, r, k)(x)_r$$
$$= xx^{n-k-1}(x)_k = x\sum S(n-1, r, k)(x)_r$$
$$= \sum S(n-1, r, k)(x)_{r+1} + \sum S(n-1, r, k)r(x)_r$$

and equating the coefficients of $(x)_r$ in the first and last expressions of this chain of equalities we get (A13). The same relation (A13) can also be obtained from (A6). Relations (A14) and (A15) are derived from (A13) and (A12) with the help of (A11). The relation (A13) is equivalent to the relation (23) for the Stirling numbers. All different matrices $S(n, r, 1)$, $S(n, r, 2)$, ... can be obtained from different initial values, for example, $S(n, 0, 1) = \delta_{n0}$; $S(0, 0, 2) = -S(1, 0, 2) = 1$, $S(n, 0, 2) = 0$, $n \geqslant 2$; $S(0, 0, 3) = 1$, $S(1, 0, 3) = -3$, $S(2, 0, 3) = 2$, $S(n, 0, 3) = 0$, $n \geqslant 3$. These different matrices can also be derived from each other by means of (A12).

(5) Exponential generating functions

$$Y_r^{(k)}(x)=\sum_n S(n+k,r,k)x^n/n!, \tag{A17}$$

$$Z_r^{(k)}(x)=\sum_n S^{-1}(n+k,r,k)x^n/n!, \tag{A18}$$

$(n=0,1,2,\ldots)$ can be found following the method of Riordan (1958). Because of coincidence of (A13) and (23) functions $Y_r^{(k)}(x)$ satisfy the same differential equation

$$(\mathrm{d}/\mathrm{d}x-r)Y_r^{(k)}(x)=Y_{r-1}^{(k)}(x) \tag{A19}$$

which is independent of k. Then we can show that

$$Y_r^{(k)}(x)=\exp(kx)(\exp x-1)^{r-k}/(r-k)! \tag{A20}$$

is the solution of (A19) when the initial condition for the equation (A19) is the function $Y_k^{(k)}(x)=\exp(kx)$ which follows from the fact that $S(n+k,k,k)=k^n$.

If we use relation (A15), we obtain the differential equation

$$(1+x)(\mathrm{d}/\mathrm{d}x)Z_r^{(k)}+kZ_r^{(k)}(x)=Z_{r-1}^{(k)}(x) \tag{A21}$$

which now depends on k and differs, therefore, from the Riordan equation. We can readily deduce the solution of (A21)

$$Z_r^{(k)}(x)=[\ln(1+x)]^{r-k}/(r-k)!(1+x)^k \tag{A22}$$

from the initial function $Z_k^{(k)}(x)=(1+x)^{-k}$ which follows from the fact that $S^{-1}(n+k,k,k)=(-1)^n(n+k-1)_n$.

(6) The following property of $S(n,r,k)$

$$\sum_r S(n,r,l)S^{-1}(r,m,k)=S(n-m+k-1,k-1,l) \tag{A23}$$

has no analogue in the case of Stirling numbers of the first and second kinds since for $l=k$, (A23) reduces to the orthogonality relation (A4). We base the proof of (A23) on the recurrence relations. Suppose that (A23) is true; if we transform the left-hand side of this equation with the help of (A14)

$$\sum_r S(n,r,l)S^{-1}(r,m,k)$$
$$=\sum S(n,r,l)S^{-1}(r,m,k-1)+(k-1)\sum S(n,r,l)S^{-1}(r,m+1,k)$$

we obtain another recurrence relation (A13) by applying (A23) to each part of this equality. To complete the proof we can point out that the initial values for the left- and right-hand sides of (A23) are equal.

Another interesting result is the extension of the definition (A3)

$$S(n,r,k)=\sum_{i=0}^{\infty} S^{-1}(k,k-i,l)S(n-i,r,l). \tag{A24}$$

To prove (A24) we multiply both sides of this equation by $S^{-1}(r,m,l)$ and carry out the sum over all r. Then we get (A23) which has already been proved. Equation (A24) may be expanded to an arbitrary number of convolutions. For example, the triple convolution is

$$S(n,r,k)=\sum_{i_1,i_2,i_3} S^{-1}(k,k-i_1,l_1)S^{-1}(l_1,l_1-i_2,l_2)S^{-1}(l_2,l_2-i_3,l_3)\times S(n-i_1-i_2-i_3,r,l_3), \qquad \text{(A25)}$$

where l_1, l_2, l_3 are arbitrary integers.

(7) Associated Stirling numbers of the second kind $b(m,j)$ (Riordan 1958)

$$S(n,r)=\sum_j \binom{n}{n-r+j} b(n-r+j,j) \qquad \text{(A26)}$$

can be generalised to our case

$$S(n,r,k)=\sum_{j=0}^{n-r}\binom{n-k}{n-r+j} b(n-r+j,j,k) \qquad \text{(A27)}$$

$$b(m,j,k)=\sum_{i=0}^{m-2j} b(2j+i,j)\binom{m}{2j+i}k^{m-2j-i} \qquad \text{(A28)}$$

are generalised associated Stirling numbers. Our proof of (A27)–(A28) is not compact enough to be produced here.

(8) In conclusion we write out a list of some particular values and checking relations which help to define generalised Stirling numbers more correctly from different points of view:

$$S(n,r,0)=S(n,r,1)=S(n,r)\text{—Stirling numbers of the second kind,} \qquad \text{(A29)}$$

$$S^{-1}(n,r,0)=S^{-1}(n,r,1)=s(n,r)\text{—Stirling numbers of the first kind,} \qquad \text{(A30)}$$

$$S(n,n,k)=S^{-1}(n,n,k)=1, \qquad \text{(A31)}$$

$$S(n,n-1,k)=-S^{-1}(n,n-1,k)=\binom{n}{2}-\binom{k}{2}, \qquad \text{(A32)}$$

$$S(n,r,k)=S^{-1}(n,r,k)=0 \qquad \text{for } r>n \qquad \text{or } r\leqslant k-1 \text{ and } n\geqslant k, \qquad \text{(A33)}$$

$$S(n,k,k)=k^{n-k}, \qquad \text{(A34)}$$

$$S^{-1}(n,k,k)=(-1)^{n-k}(n-1)_{n-k}, \qquad \text{(A35)}$$

$$\sum_{r=0}^{n}(-1)^{n-r}r!\,S(n,r,k)=\begin{cases} k!, & n=k-1,k,k+1,\ldots, \\ \sum_{j=k-n}^{k}(-1)^{k-j}s(k,j), & n=0,1,\ldots,k-2, \end{cases} \qquad \text{(A36)}$$

$$\sum_r (-1)^{n-r}S^{-1}(n,r,k)=n!/k!, \qquad n\geqslant k, \qquad \text{(A37)}$$

$$\sum_r S^{-1}(n,r,k)=(-1)^{n-k}(n-2)!/(k-2)!, \qquad n\geqslant k, \qquad \text{(A38)}$$

$$\sum_{n=m}^{r} S(n-1,r-1,k)r^{m-n}=S(m,r,k), \qquad \text{(A39)}$$

$$\sum_{r=m}^{n} S^{-1}(n+1,r+1,k)n^{r-m}=S^{-1}(n,m,k). \qquad \text{(A40)}$$

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